Ch: Set & Seq
part 2

Ordered Field

A field S is an ordered field if it satisfies the following:

(1) Law of Tricholomy
(2) Transitivity
(3) Compatibility with addition
(4) Compatibility with multiplication

(1) Law of Trichotomy:

for $a, b \in S$ any one is true

$a > b$, $a = b$, $a < b$

(2) Transitivity: $a > b$, $b > c \Rightarrow a > c$

(3) Compatibility with addition:

$a > b \Rightarrow a + c > b + c$

(4) " " multiplication.

$a > b$ & $c > 0 \Rightarrow ac > bc$

Properties of $\mathbb{N}$

Well ordering Property:

Every non-empty subset of $\mathbb{N}$ has a least element

Example :

$S = \{2, 5, 7\}$ ← non empty subset of $\mathbb{N}$, $\alpha = 5$ (Take any $\alpha \in S$)

(upto α) $S_1 = \{2, 5\}$ is a finite subset of $\mathbb{N}$

$\Rightarrow$ it has a least element β

$\Rightarrow \quad 1 \leq \beta \leq 5$

Let $\gamma \in S \Rightarrow$ either $\gamma > 5$ or $\gamma \leq 5$

Now, $\gamma > 5$
$\Rightarrow \beta < \gamma$
(no problem)

$\gamma \leq 5 \Rightarrow \gamma \in S_1 \Rightarrow \beta \leq \gamma \Rightarrow \beta$ least in S

Theorem: (Principle of Induction)

A non-empty set S $(\subseteq \mathbb{N})$ such that

(i) $1 \in S$

(ii) $k \in S \Rightarrow k+1 \in S$

Then $S = \mathbb{N}$

Proof: Let $S_1 = \mathbb{N} \setminus S$. To show $S_1 = \phi$

Indeed, if $S_1 \neq \phi$ then α (least of S_1) $\Rightarrow$ $\in S_1$ and $\alpha \neq 1$

Now, $1 \in S \Rightarrow \alpha - 1 \in S \Rightarrow \alpha \in S$

H.W. (1) Show that $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer

(2) $n! > 2^{n-1}$ for $n \geq 3$

(3) $7^n - 3^n$ is divisible by 4

☆ (4) If $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2+u_n}$ for all $n \in \mathbb{N}$ then show:

$u_n < 2$ for all $n \in \mathbb{N}$

H.W.

(1) Show that, the set of integers $\mathbb{Z}$ is not a field although it satisfies the order structure.

(2) the set of rationals $\mathbb{Q}$ is a field and satisfies order properties

Bounded (bdd) & unbounded set

Bdd above : $S \subseteq \mathbb{R}$ is bdd above if $\exists$ $B_a \in \mathbb{R}$ s.t $a \leq B_a$ $\forall a \in S$

B_a = upper bound

Bdd below ; Similar defn

Bounded ; Both bdd above & below

Least upper bound (LUB)
= supremum (sup)

Greatest Lower bound (GLB)
= infimum (inf)

Observe that:

sup & inf are unique &
may or may not belong to the set

Ex :

$$S = \{2, 4, 7\}$$

Many upper bound : 7, 8, 10, etc

" Lower " : 2, 0, -1, -12 etc

$\sup(S) = 7$ and $\inf(S) = 2$

H.W : True / False :

(1) $\sup(\mathbb{N})$ exists (2) $\inf(\mathbb{N}) = 1$

H.W:

(1) $S = \left\{ \frac{1}{n+1} : n \in \mathbb{N} \right\}$ has sup & inf

(2) Find $\sup(S)$ and $\inf(S)$

(i) $S = \left\{ \frac{n^2}{n^2+1} : n \in \mathbb{N} \right\}$

(ii) $S = \left\{ |x| : x^2 < 1, x \in \mathbb{R} \right\}$

(iii) $S = \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{R} \right\}$

(iv) $S = \left\{ \frac{(-1)^m}{m} + \frac{(-1)^n}{n} : m, n \in \mathbb{R} \right\}$

☆ For two non-empty sets A, B show that

(i) $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$

(ii) $\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$

Real Number

field ✓, ordered field ✓

* Completeness property
  ($\mathbb{R}$ satisfies, $\mathbb{Q}$ not)

* Archimedean property
  (both $\mathbb{R}$, $\mathbb{Q}$ satisfies)

If $\alpha^2 < 2 \Rightarrow$ can get a $\Rightarrow\Leftarrow$

★ Example $\alpha < \frac{4+3\alpha}{3+2\alpha} < \sqrt{2}$

$\sqrt{2}$

Let $S = \{ a \in \mathbb{Q} : a > 0, \ a^2 < 2 \}$ $a < \sqrt{2}$

$\Rightarrow$ S non empty, bdd above by 2

BUT, S has $\underline{\underline{\text{no}}}$ sup in $\mathbb{Q}$ $\Rightarrow\Leftarrow$

i.e. $\sup(S) \notin \mathbb{Q}$ (proof ?)

Hint. If $\sup(S) = \alpha$ then $\alpha^2 < 2$ or $= 2$ or > 2

1.4

1.44

Archimedean Property

If $a, b (>0) \in \mathbb{R}$ then $\exists\, n \in \mathbb{N}$ such that $nb > a$

Ex: If $a \in \mathbb{R}$ then $\exists\, n \in \mathbb{N}$ s.t. $n > a$

(Hint: $a \leq 0 \Rightarrow n = 1$; If $a > 0$ take $b = 1$)

H.W. If $a \in \mathbb{R}$, $a > 0$ then show that $\exists\, n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < a$

Sets in $\mathbb{R}$

1) open interval $\{x : a < x < b\}$

2) close " $\{x : a \le x \le b\}$

3) semi open | semi close $\{x : a < x \le b\}$ or $\{x : a \le x < b\}$

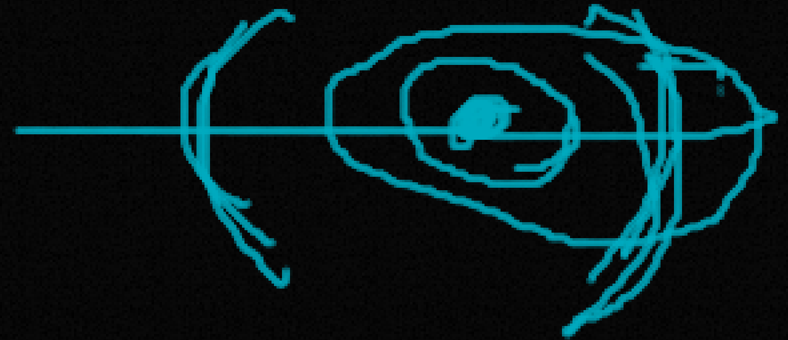
$B(a,r)$

4) Neighbourhood (nbd) : for $\alpha \in (a,b)$

$N(\alpha,\delta) \equiv (\alpha-\delta, \alpha+\delta)$ is δ-nbd of α

5) Interior point : point containg nbd

Open set :



A set S is open set if **each** point of S is an interior point of S.

eg : ✗ (2,3] ✓ (2,3)

3 is not interior pt.

Deleted nbd : $N(\alpha)\setminus\{\alpha\} \equiv N'(\alpha)$

Limit point / Cluster point :

A point $x \in S$ is said to be a limit point of S if $N'(x,\delta)\cap S \neq \phi$

$2-\frac{1}{n}, \; n \in \mathbb{N}$

eg: $S = (2,3]$ each pt of S is limit pt. Even if $2 \notin S$, but 2 is limit pt.

Isolated point : $(\exists \text{ nbd}) \cap S = \phi$ $\{1, 2, 3, \ldots\}$ $(1, \frac{1}{2})$ $(1, 5)$

$1 \cup [2, 3]$ $y \in S$ is isolated point of S
if y is not limit pt of S.

eg: $\mathbb{N}$ has no limit pt, all isolate

Bolzano-Weierstrass Theorem:

Every bdd infinite subset of $\mathbb{R}$ has atleast one limit pt in $\mathbb{R}$